

The initial mass function modeled by a left truncated beta distribution

Lorenzo Zaninetti

*Dipartimento di Fisica, Via Pietro Giuria 1,
10125 Torino, Italy*

zaninetti@ph.unito.it
http://www.ph.unito.it/~zaninett

ABSTRACT

The initial mass function (IMF) for the stars is usually fitted by three straight lines, which means seven parameters. The presence of brown dwarfs (BD) increases to four the straight lines and to nine the parameters. Another common fitting function is the lognormal distribution, which is characterized by two parameters. This paper is devoted to demonstrating the advantage of introducing a left truncated beta probability density function, which is characterized by four parameters. The constant of normalization, the mean, the mode and the distribution function are calculated for the left truncated beta distribution. The normal-beta (NB) distribution which results from convolving independent normally distributed and beta distributed components is also derived. The chi-square test and the K-S test are performed on a first sample of stars and BDs which belongs to the massive young cluster NGC 6611 and on a second sample which represents the star's masses of the cluster NGC 2362.

Subject headings: Stars: luminosity function, mass function; Stars: fundamental parameters; Methods: statistical

1. Introduction

The distribution in mass of the stars has been fitted with a power law starting with Salpeter (1955). He suggested $\xi(m) \propto m^{-\alpha}$ where $\xi(m)$ represents the probability of having a mass between m and $m + dm$; He found $\alpha = 2.35$ in the range $10M_{\odot} > M \geq 1M_{\odot}$; this value has changed little with time and a recent evaluation quotes 2.3, see Kroupa (2001). Subsequent research has started to analyze the initial mass function (IMF) with three power

laws, see Scalo (1986); Kroupa et al. (1993); Binney & Merrifield (1998) and four power laws, see Kroupa (2001). A first comment on this temporal evolution is that the name is not appropriate because the power function distribution $\xi(m) \propto m^b$ is defined only for positive values of b , see Evans et al. (2000). A second comment is that the exact name for a probability density function (PDF) $\xi(m) \propto m^{-(c+1)}$ with $c > 0$ is the Pareto distribution. A third comment is that this progressive increase in the number of segments has limited the development of new or modified PDFs. The approach to the IMF by a continuous distribution has been modeled by the lognormal distribution in order to fit both the range of the stars and the brown dwarfs (BDs) regime, see Chabrier (2003). Recall that usually the standard PDFs such as the lognormal, gamma, generalized gamma, and Weibull are defined in the interval $0 \leq x < \infty$. The fact that the number of stars with mass $m < 0.07M_{\odot}$ is nearly zero suggests a left truncated PDF. Our analysis has therefore been focused on the beta distribution, which by definition has an upper bound. From the previous analysis the following questions can be raised.

- Is it possible to find the constant of normalization for a left truncated beta PDF?
- Is it possible to derive an analytical expression for the mean, the mode, and the distribution function of a left truncated beta PDF?
- Is a left truncated beta PDF an acceptable model for the IMF as well as a real sample of masses?

In order to answer the previous questions, we first review some standard PDFs, see Section 2. We subsequently introduce the various beta PDFs, the convolution of a beta PDF with a normal PDF, and a left truncated beta, see Section 3. In order to find out which PDF performs best, the two main criteria which report the goodness of fit are found in Section 4. A comparison between various continuous PDFs and the left truncated beta is carried out in Sections 5.2 and 5.3 for two samples of stars.

2. Distributions commonly used

This section reviews some standard PDFs, namely, the lognormal, gamma, generalized gamma, Pareto, truncated Pareto, and the recently developed Double Pareto-lognormal distribution.

2.1. Lognormal distribution

Let X be a random variable taking values x in the interval $[0, \infty]$; the *lognormal* PDF , following Evans et al. (2000) or formula (14.2)' in Johnson et al. (1994), is

$$f_{LN} = \frac{1}{x\sigma\sqrt{2\pi}} \exp \frac{-[\ln(x/m)]^2}{2\sigma^2} \quad , \quad (1)$$

or

$$f_{LN} = \frac{1}{x\sigma\sqrt{2\pi}} \exp \frac{-(\ln x - \mu_{LN})^2}{2\sigma^2} \quad , \quad (2)$$

where $m = \exp \mu_{LN}$ and $\mu_{LN} = \log m$.

2.2. Gamma distribution

Let X be a random variable taking values x in the interval $[0, \infty]$; the *gamma* PDF is

$$p(x; b, c) = \frac{\left(\frac{x}{b}\right)^{c-1} e^{-\frac{x}{b}}}{b \Gamma(c)} \quad , \quad (3)$$

where $\Gamma(z)$ is the gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad , \quad (4)$$

see formula (17.1) in Johnson et al. (1994).

2.3. Generalized gamma distribution

Let X be a random variable taking values x in the interval $[0, \infty]$; the *generalized gamma* PDF, following Evans et al. (2000), is

$$f(x; a, b, c) = c \frac{b^{a/c}}{\Gamma(a/c)} x^{a-1} \exp(-bx^c) \quad , \quad (5)$$

see formula (17.116) in Johnson et al. (1994).

2.4. The Pareto and the truncated Pareto distributions

Let X be a random variable taking values x in the interval $[a, \infty]$, $a > 0$. The *Pareto* PDF is defined by

$$f(x; a, c) = ca^c x^{-(c+1)} \quad , \quad (6)$$

with $c > 0$, see formula (20.3) in Johnson et al. (1994). The traditional Salpeter slope is therefore $-(c+1)$. An upper truncated Pareto random variable is defined in the interval $[a, b]$ and the corresponding PDF, following

Goldstein et al. (2004); Aban et al. (2006); Zaninetti & Ferraro (2008); White et al. (2008), is

$$f_T(x; a, b, c) = \frac{ca^c x^{-(c+1)}}{1 - \left(\frac{a}{b}\right)^c} . \quad (7)$$

Their means are

$$E(x; a, c) = \frac{ac}{c-1} \quad (8)$$

and

$$E(x; a, b, c)_T = \frac{ca \left(-1 + \left(\frac{a}{b}\right)^{c-1}\right)}{(c-1) \left(-1 + \left(\frac{a}{b}\right)^c\right)} . \quad (9)$$

2.5. The double Pareto-lognormal distribution

The double Pareto lognormal distribution has been recently derived, see formula (22) in Reed & Jorgensen (2004), and has been used to fit the actual sizes of cities, see Giesen et al. (2010); its PDF is

$$f(x; \alpha, \beta, \mu, \sigma) = 1/2 \alpha \beta \left(e^{1/2 \alpha (\alpha \sigma^2 + 2\mu - 2 \ln(x))} \operatorname{erfc}\left(1/2 \frac{(\alpha \sigma^2 + \mu - \ln(x))\sqrt{2}}{\sigma}\right) + e^{1/2 \beta (\beta \sigma^2 - 2\mu + 2 \ln(x))} \operatorname{erfc}\left(1/2 \frac{(\beta \sigma^2 - \mu + \ln(x))\sqrt{2}}{\sigma}\right) \right) x^{-1} (\alpha + \beta)^{-1}, \quad (10)$$

where α and β are the Pareto coefficients for the upper and the lower tail, respectively, μ and σ are the lognormal body parameters, and erfc is the complementary error function. The parameters can be found minimizing the maximum distance, D , of the K-S test, see Sect. 4.

3. Various Beta distributions

This section reviews the beta PDF defined in $[0, 1]$, the beta with scale PDF defined in $[0, b]$ and the general beta defined in $[a, b]$. The left truncated beta PDF defined in $[a, b]$ but with a finite value of probability at $x = a$ is explored. The convolution of a beta distribution with a normal distribution is also discussed.

3.1. Beta distribution

Let X be a random variable taking values x in the interval $[0, 1]$; the *beta* PDF is

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha, \beta)} \quad , \quad (11)$$

with $\alpha > 0$ and $\beta > 0$, see Evans et al. (2000) or formula (25.2) in Johnson et al. (1995). Here B is the beta function defined by

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \quad . \quad (12)$$

Its mean is

$$E(x; \alpha, \beta) = \frac{\alpha}{\alpha + \beta} \quad , \quad (13)$$

and its variance,

$$\sigma^2(x; \alpha, \beta) = \frac{\alpha \beta}{(1 + \alpha + \beta) (\alpha + \beta)^2} \quad , \quad (14)$$

see formula (25.15a) in Johnson et al. (1995). The mode is at

$$m(x; \alpha, \beta) = \frac{\alpha - 1}{\alpha - 2 + \beta} \quad . \quad (15)$$

The method of matching moments gives the following parameter estimation

$$\tilde{\alpha} = \bar{x} \left(\frac{\bar{x} (1 - \bar{x})}{s^2} - 1 \right) \quad , \quad (16)$$

and

$$\tilde{\beta} = (1 - \bar{x}) \left(\frac{\bar{x} (1 - \bar{x})}{s^2} - 1 \right) \quad , \quad (17)$$

where \bar{x} and s^2 are the mean and the variance of the sample. The distribution function (DF) is

$$DF(x; \alpha, \beta) = \frac{x^\alpha {}_2F_1(\alpha, -\beta + 1; \alpha + 1; x)}{\beta (\alpha, \beta) \alpha} \quad , \quad (18)$$

where ${}_2F_1(a, b; c; z)$ is the regularized hypergeometric function Abramowitz & Stegun (1965); von Seggern (1992); Thompson (1997); Gradshteyn, I. S. and Ryzhik, I. M. and Jeffrey, A. and Zwillinger, D. (2007); Olver et al. (2010).

3.2. Beta distribution with scale

Let X be a random variable taking values x in the interval $[0, b]$. The *beta with scale* PDF is

$$f_b(x; b, \alpha, \beta) = \frac{\left(\frac{x}{b}\right)^{\alpha-1} \left(1 - \frac{x}{b}\right)^{\beta-1}}{B(\alpha, \beta) b}. \quad (19)$$

Its expected mean is

$$E(x; b, \alpha, \beta) = \frac{\alpha b}{\alpha + \beta}, \quad (20)$$

and its variance,

$$\sigma(x; b, \alpha, \beta)_b^2 = \frac{\beta \alpha b^2}{(1 + \alpha + \beta)(\alpha + \beta)^2}. \quad (21)$$

The mode is at

$$m(x; b, \alpha, \beta)_b = \frac{b(\alpha - 1)}{\beta - 2 + \alpha}. \quad (22)$$

The DF is

$$DF_b(x; b, \alpha, \beta) = \frac{x^\alpha {}_2F_1(\alpha, -\beta + 1; \alpha + 1; x)}{\beta(\alpha, \beta) \alpha}. \quad (23)$$

The three parameters can be estimated by

$$\tilde{b} = \text{maximum of sample}, \quad (24)$$

$$\tilde{\alpha} = -\frac{\bar{x}(-b\bar{x} + \bar{x}^2 + s^2)}{s^2 b}, \quad (25)$$

$$\tilde{\beta} = -\frac{(b - \bar{x})(-b\bar{x} + \bar{x}^2 + s^2)}{s^2 b}. \quad (26)$$

3.3. General Beta distribution

Let X be a random variable taking values x in the interval $[a, b]$. The *general beta* PDF is

$$f_{ab}(x; a, b, \alpha, \beta) = \frac{(b - a)(x - a)^{\alpha-1}(b - x)^{\beta-1}}{b^{\alpha+\beta-1} b \left(\frac{b-a}{b}\right)^{\alpha+\beta} B(\alpha, \beta)}, \quad (27)$$

see formula (25.1) in Johnson et al. (1995). Its expected mean is

$$E(x; a, b, \alpha, \beta)_{ab} = \frac{\alpha b + a\beta}{\alpha + \beta}, \quad (28)$$

and its variance,

$$\sigma(x; a, b, \alpha, \beta)_{ab}^2 = \frac{(a-b)^2 \alpha \beta}{(\alpha + \beta + 1)(\alpha + \beta)^2} . \quad (29)$$

The mode is at

$$m(x; a, b, \alpha, \beta)_{ab} = \frac{\alpha b - b + a\beta - a}{-2 + \alpha + \beta} . \quad (30)$$

The four parameters can be estimated by

$$\tilde{a} = \text{minimum of sample} \quad \tilde{b} = \text{maximum of sample} , \quad (31)$$

$$\tilde{\alpha} = -\frac{(-\bar{x} + \tilde{a}) \left(-\bar{x} \tilde{b} + \tilde{b} \tilde{a} + \bar{x}^2 + s^2 - \tilde{a} \bar{x} \right)}{s^2 \left(\tilde{a} - \tilde{b} \right)} , \quad (32)$$

$$\tilde{\beta} = \frac{\left(\tilde{b} - \bar{x} \right) \left(-\bar{x} \tilde{b} + \tilde{b} \tilde{a} + \bar{x}^2 + s^2 - \tilde{a} \bar{x} \right)}{s^2 \left(\tilde{a} - \tilde{b} \right)} , \quad (33)$$

see the discussion in section 25.4 of Johnson et al. (1995).

3.4. Left truncated beta distribution with scale

Let X be a random variable taking values x in the interval $[a, b]$ and having a finite value in a . The *left truncated beta with scale* PDF is

$$f_T(x; a, b, \alpha, \beta) = K x^{\alpha-1} (b-x)^{\beta-1} , \quad (34)$$

where the constant is

$$K = \frac{-\alpha \Gamma(\alpha + \beta)}{b^{\beta-1} H a^\alpha \Gamma(\alpha + \beta) - b^{\beta-1+\alpha} \Gamma(1 + \alpha) \Gamma(\beta)} , \quad (35)$$

and

$$H = {}_2F_1(\alpha, -\beta + 1; 1 + \alpha; \frac{a}{b}) . \quad (36)$$

The constant of normalization can be obtained from the integral of the beta with scale PDF as represented by Equation (19). The integral 3.194.1 on p315 of Gradshteyn, I. S. and Ryzhik, I. M. and Jeffrey, A. and Zwillinger, D. (2007),

$$\begin{aligned} & \int_0^u x^{\mu-1} (1 + \beta x)^\nu dx \\ &= \frac{e^{\mu \ln(u)} {}_2F_1(\mu, -\nu; 1 + \mu; -\beta u)}{\mu} , \end{aligned} \quad (37)$$

is useful in the analytical derivation of the constant. This PDF, which is new and therefore cannot be found in Johnson et al. (1995), at $x = a$ is not zero but takes the finite value

$$f_T(a; a, b, \alpha, \beta) = K a^{\alpha-1} (b-a)^{\beta-1} . \quad (38)$$

Its expected mean is

$$E(x; a, b, \alpha, \beta)_T = K \left(\frac{b^{\beta-1} b^{1+\alpha} {}_2F_1(-\beta+1, 1+\alpha; 2+\alpha; 1)}{1+\alpha} - \frac{b^{\beta-1} a^{1+\alpha} {}_2F_1(-\beta+1, 1+\alpha; 2+\alpha; \frac{a}{b})}{1+\alpha} \right) . \quad (39)$$

The mode is at

$$m(x; b, \alpha, \beta)_T = \frac{b(\alpha-1)}{\alpha-2+\beta} , \quad (40)$$

and in order to exist one must have $m_T > a$. The DF is

$$\begin{aligned} DF_T(x; a, b, \alpha, \beta) = \\ \Gamma(\alpha+\beta) x^\alpha {}_2F_1(\alpha, -\beta+1; 1+\alpha; \frac{x}{b}) H_D^{-1} \\ - {}_2F_1(\alpha, -\beta+1; 1+\alpha; \frac{a}{b}) a^\alpha \Gamma(\alpha+\beta) H_D^{-1} , \end{aligned} \quad (41)$$

where

$$H_D = {}_2F_1(\alpha, -\beta+1; 1+\alpha; \frac{a}{b}) a^\alpha \Gamma(\alpha+\beta) - \Gamma(\beta) b^\alpha \Gamma(\alpha) \alpha . \quad (42)$$

The survival function is

$$S_T(x; a, b, \alpha, \beta) = 1 - DF_T(x; a, b, \alpha, \beta) . \quad (43)$$

The four parameters can be estimated by

$$\tilde{a} = \text{minimum of sample} \quad \tilde{b} = \text{maximum of sample} . \quad (44)$$

A first couple for $\tilde{\alpha}$ and $\tilde{\beta}$ can be obtained from those of the beta distribution with scale as given by Equations (25) and (26). A subsequent numerical loop around the previous values gives the couple which minimize the χ^2 .

3.5. Beta distribution + normal

We consider the sum $Z = X + Y$ where X is a standard normal random variable, $N(x; \sigma)$, and Y is a general beta distribution, $f_{ab}(y; a, b, \alpha, \beta)$, as represented by the PDF (27). The sum, $NB(z; a, b, \alpha, \beta, \sigma)$, is

$$NB(z; a, b, \alpha, \beta, \sigma) = \int_a^b N(z - y) f_{ab}(y; a, b, \alpha, \beta) dy \quad . \quad (45)$$

A similar example, uniform + normal, can be found in (Brandt & Gowan 1998, Sec. 5.11.2).

This integral has an analytical solution for α and β integers, for example, when $\alpha=1$ and $\beta=1$,

$$NB(z; a, b, 1, 1, \sigma) = 1/2 \operatorname{erf} \left(1/2 \frac{\sqrt{2}a}{\sigma} - 1/2 \frac{\sqrt{2}z}{\sigma} \right) (a - b)^{-1} - 1/2 \operatorname{erf} \left(1/2 \frac{\sqrt{2}b}{\sigma} - 1/2 \frac{\sqrt{2}z}{\sigma} \right) (a - b)^{-1} \quad , \quad (46)$$

where erf is the error function.

4. Goodness of fit tests

The occasional reader may question which is the best fit for the distributions analyzed here. In order to answer this question, we first introduce χ^2 , which is computed according to the formula

$$\chi^2 = \sum_{i=1}^n \frac{(T_i - O_i)^2}{T_i} \quad , \quad (47)$$

where n is the number of bins, T_i is the theoretical value, and O_i is the experimental value represented by the frequencies. The theoretical frequency distribution is given by:

$$T_i = N \Delta x_i p(x) \quad , \quad (48)$$

where N is the number of elements of the sample, Δx_i is the magnitude of the size interval, and $p(x)$ is the PDF under examination. The size of the bins, Δx_i , is equal for each bin in the the case of linear histograms, but different for each bin when logarithmic histograms are considered.

A reduced merit function χ_{red}^2 is evaluated by

$$\chi_{red}^2 = \chi^2 / NF \quad , \quad (49)$$

where $NF = n - k$ is the number of degrees of freedom, n is the number of bins, and k is the number of parameters. The goodness of the fit can be expressed by the probability Q , see equation 15.2.12 in Press et al. (1992), which involves the degrees of freedom and the χ^2 . According to Press et al. (1992), the fit “may be acceptable” if $Q > 0.001$. The Akaike information criterion (AIC), see Akaike (1974), is defined by

$$AIC = 2k - 2\ln(L) \quad , \quad (50)$$

where L is the likelihood function and k the number of free parameters in the model. We assume a Gaussian distribution for the errors and the likelihood function can be derived from the χ^2 statistic $L \propto \exp(-\frac{\chi^2}{2})$ where χ^2 has been computed by Equation (47), see Liddle (2004), Godlowski & Szydowski (2005). Now the AIC becomes

$$AIC = 2k + \chi^2 \quad . \quad (51)$$

We also perform the Kolmogorov–Smirnov test (K-S), see Kolmogoroff (1941); Smirnov (1948); Massey (1951), which does not require binning the data. The K-S test, as implemented by the FORTRAN subroutine KSONE in Press et al. (1992), finds the maximum distance, D , between the theoretical and the astronomical DF as well the significance level P_{KS} , see formulas 14.3.5 and 14.3.9 in Press et al. (1992). Values of $P_{KS} \geq 0.1$ assures that the fit is acceptable.

5. Astrophysical applications

This section reviews the galactic IMF as modeled by three and four power laws PDFs and fits the masses of the cluster NGC 2362 and the cluster NGC 6611 with the various PDFs here considered.

5.1. Galactic IMF

The IMF is usually modeled by two or three power laws of the type

$$p_{stars}(m) \propto x^{-\alpha_i} \quad , \quad (52)$$

each zone being characterized by a different exponent α_i . In order to have a PDF normalized to unity, one must have

$$\sum_{i=1,3} \int_{m_i}^{m_{i+1}} c_i m^{-\alpha_i} dm = 1 \quad . \quad (53)$$

For example, we start with $c_1=1$: c_2 will be determined by the following equation

$$c_1(0.5 - \epsilon)^{-\alpha_1} = c_2(0.5 + \epsilon)^{-\alpha_2} \quad , \quad (54)$$

where ϵ is a small number, e.g., $\epsilon = 10^{-4}$. In the previous equation we insert $\alpha_1 = 1.3$ and $\alpha_2 = 2.3$ and therefore $c_2 = 0.503$. The same procedure applied to c_3 gives $c_3 = 0.506$. The integral of $p_{stars}(m)$ over the field of existence now gives 4.14, but according to the requirement of normalization as given by Equation (53), it should be 1. As a consequence, the three constants are now $c_1 = 0.24$, $c_2 = 0.1205$, and $c_3 = 0.1206$, which is the same as equation (59) in Kroupa et al. (2012)

$$p(m) = \begin{cases} 0.24 x^{-1.3} & \text{if } 0.07M_{\odot} < m \leq 0.5M_{\odot} \\ 0.12 x^{-2.3} & \text{if } 0.5M_{\odot} < m \leq 1.0M_{\odot} \\ 0.12 x^{-2.7} & \text{if } 1.0M_{\odot} < m \leq 10M_{\odot} \end{cases} \quad . \quad (55)$$

The mean of the galactic IMF is given by a numerical integration over the three zones

$$\bar{m} = \sum_{i=1,3} \int_{m_i}^{m_{i+1}} c_i m m^{-\alpha_i} dm = 0.389M_{\odot} \quad . \quad (56)$$

The presence of the brown dwarfs means the use of four power laws instead of three power laws:

$$p(m) = \begin{cases} 2.194 x^{-0.3} & \text{if } 0.01M_{\odot} < m \leq 0.07M_{\odot} \\ 0.153 x^{-1.3} & \text{if } 0.07M_{\odot} < m \leq 0.5M_{\odot} \\ 0.076 x^{-2.3} & \text{if } 0.5M_{\odot} < m \leq 1.0M_{\odot} \\ 0.076 x^{-2.7} & \text{if } 1.0M_{\odot} < m \leq 10M_{\odot} \end{cases} \quad , \quad (57)$$

where in order to have a continuous PDF, the BDs have the range $0.01M_{\odot} < m \leq 0.07M_{\odot}$ rather than $0.01M_{\odot} < m \leq 0.15M_{\odot}$, see equation (59) in Kroupa et al. (2012). We have covered the galactic four power laws, we now introduce the generalized four power laws $p_G(m; -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, m_1, m_2, m_3, m_4, m_5)$ which in the case of NGC 2362 is

$$p_G(m; -0.01, -0.02, -1.1, -2.7, 0.01, 0.07, 0.50, 1.0, 10.) \quad \text{NGC 2362 case} \quad , \quad (58)$$

and in the case of NGC 6611 is

$$p_G(m; -0.01, -0.6, -2.4, -2.7, 0.01, 0.07, 0.50, 1.0, 10.) \quad \text{NGC 6611 case} \quad . \quad (59)$$

5.2. IMF of NGC 2362

A photometric survey of NGC 2362 allows of deducing the mass of 271 stars in the range $1.47M_{\odot} > M \geq 0.11M_{\odot}$, see Irwin et al. (2008) and the data in J/MNRAS/384/675 at

the Centre de Donns astronomiques de Strasbourg (CDS). Table 1 shows the values of χ_{red}^2 , the AIC, the probability Q , of the astrophysical fits and the results of the K-S test.

Table 1: Numerical values of χ_{red}^2 , AIC, probability Q , D, the maximum distance between theoretical and observed DF, and P_{KS} , significance level, in the K-S test for the mass distribution of the NGC 2362 cluster data (272 stars). The number of linear bins, n , is 20.

PDF	parameters	AIC	χ_{red}^2	Q	D	P_{KS}
lognormal	$\sigma=0.5, \mu_{LN} = -0.55$	37.64	1.86	0.013	0.07305	0.10486
double Pareto-lognormal	$\sigma=0.44, \mu_{LN} = -0.52$ $\alpha=5, \beta = 5$	40.42	2.02	0.008	0.066103	0.17882
general beta	$a = 0.12, b = 1.47$ $\alpha = 1.67, \beta = 2.77$	29.09	1.31	0.17	0.059141	0.288813
general beta +normal (NB)	$a = 0.12, b = 1.47$ $\alpha = 1.67, \beta = 2.77, \sigma = 0.001$	31.09	1.40	0.13	0.06412	0.20612
left truncated beta	$a = 0.12, b = 1.47$ $\alpha = 2.23, \beta = 3.09$	31.19	1.44	0.1	0.06158	0.24572
four power laws	Eqn. (58)	77.608	4.89	$1.17 \cdot 10^{-8}$	0.16941	$2.60363 \cdot 10^{-7}$

Figure 1 shows the fit with the left truncated beta distribution of NGC 2362 and Figure 2 visually compares the four types of fits for NGC 2362.

5.3. IMF of NGC 6611

The massive young cluster NGC 6611 has been carefully analyzed from the point view of the IMF in the range $1.5M_{\odot} > M \geq 0.02M_{\odot}$. This means that also the BD range is covered, see more details in Oliveira et al. (2009) with data in J/MNRAS/392/1034 at the CDS. Figure 3 shows the fit with the left truncated beta distribution of NGC 6611 and Figure 4 shows a visual comparison of four types of fits for NGC 6611. Table 2 shows the values of χ_{red}^2 , the AIC, and the probability Q of the astrophysical fits and the results of the K-S test. Figure 5 shows the residuals and χ^2 as a function of the middle value of the logarithmic bin considered, both for the left truncated beta and for the lognormal.

Table 2: Numerical values of χ_{red}^2 , AIC, probability Q , D, the maximum distance between theoretical and observed DF, and P_{KS} , significance level, in the K-S test for the mass distribution of NGC 6611 cluster data (207 stars + BDs). The number of linear bins, n , is 20.

PDF	parameters	AIC	χ_{red}^2	Q	D	P_{KS}
lognormal	$\sigma=1.029, \mu_{LN} = -1.258$	71.24	3.73	$1.3 \cdot 10^{-7}$	0.09366	0.04959
double Pareto-lognormal	$\sigma= 0.979, \mu_{LN} = -1.208$ $\alpha=4, \beta = 4$	70.3	3.89	$2.13 \cdot 10^{-7}$	0.07995	0.13523
general beta	$a = 0.019, b=1.46$ $\alpha = 0.56, \beta=1.55$	39.29	1.956	0.0123	0.11456	0.007924
general beta +normal (NB)	$a = 0.019, b=1.46$ $\alpha = 0.56, \beta=1.55, \sigma = 0.001$	41.3	2.08	0.008	0.09476	0.04545
left truncated beta	$a = 0.019, b=1.46$ $\alpha = 0.55, \beta=1.6$	42.09	2.13	0.005	0.06839	0.27781
four power laws	Eqn. (59)	81.39	5.18	$2.41 \cdot 10^{-9}$	0.12514	$2.7239 \cdot 10^{-3}$

6. Conclusions

Motivations In the last 50 years, the IMF has been modeled progressively by one power law, by two power laws, three power laws, and four power laws. The three power law distribution has seven parameters and the four power law has nine, and they both have a finite range of existence. A second widely used fitting function is the lognormal, which is characterized by two parameters and is defined on the interval $[0, \infty]$. In this paper, we have described a left truncated beta PDF which has (i) a lower and an upper bound, (ii) a finite value of probability on the lower bound rather than zero, (iii) two parameters, α and β , which fix the shape of the distribution, (iv) an analytical expression for the average value. Two physical meanings are distinguished: (i) the upper limit of the left truncated beta is connected with the maximum stellar mass, which is $\approx 60M_{\odot}$, (ii) the lower limit is connected with an unknown physical mechanism which limits the distribution in masses. Further on we remember that the lognormal PDF has the important disadvantage of missing the well-accepted Salpeter-type high-mass power law. The high masses behavior of the various PDFs here analyzed is reported in Figure 6 for NGC 6611 where the Pareto and truncated Pareto PDFs are evaluated for $M \geq 0.43M_{\odot}$, which means Salpeter slope -2.3. From the previous Figure the discrepancy of the lognormal and double Pareto-lognormal at

high masses is evident.

Goodness of fit tests The statistical tests here performed are split in two: (i) a first test requires binning the data in order to evaluate χ^2 , and the indicators are χ_{red}^2 , the AIC, and the probability Q ; (ii) the K-S test does not require binning the data, and the two indicators are D and P_{KS} . These two tests, when applied to NGC2362 and NGC6611, indicate that the beta family (general and left truncated) performs better than the lognormal distribution both when the binning of the data is computed, see Tables example, the K-S test for the mass distribution of NGC6611 indicates a confidence level of 27% for the left truncated beta and 5% for the lognormal. New confidence levels can be found with the Anderson-Darling test which is a modification of the K-S test, see Stephens (1974) and the discussion at <https://asaip.psu.edu/Articles/beware-the-kolmogorov-smirnov-test>. Currently tables of critical values for the Anderson-Darling test are available for the lognormal PDF but the critical values for other PDFs here explored are not yet available, see <http://www.itl.nist.gov/div898/handbook>.

Convolution The random sum (convolution) of a general beta and a normal random variable, as represented by Equation (45), when applied to NGC6611 introduces a further parameter, σ , which increases χ_{red}^2 and the AIC from that of the general beta, but decreases D and P_{KS} in the K-S test, see Table 2.

The mode A careful attention should be paid to the falloff of the IMF towards the brown dwarfs. The left truncated beta PDF, see PDF (34), once the numbers of the open cluster NGC6611 are inserted, see Figure 3, decreases after the maximum at $m \approx 0.019M_{\odot}$. This fact can be explained by the following Taylor expansion

$$f_T(x; 0.019, 1.46, 0.715, 2.185) = 2.65 - 21.64 (x - 0.039) + 336.35 (x - 0.039)^2 + O((x - 0.039)^3) \quad . \quad (60)$$

The previous decreasing function converts itself into an increasing function when the integration is performed

$$\int_{0.019}^x f_T(x; 0.019, 1.46, 0.715, 2.185) = 3.50 x - 10.82 x^2 + 112.11 (x - 0.039)^3 \quad , \quad (61)$$

and we recall that the evaluation of the frequencies corresponds to an integration.

Lognormal family The recently formulated double Pareto-lognormal distribution draws attention to a possible alternative to the lognormal. Our tests show that the double Pareto-lognormal lowers the value of the maximum distance, D , of the K-S test, see Tables 1 and

2. Inconveniently, at the moment of writing there are no analytical evaluations of the four parameters which characterize the double Pareto-lognormal.

The astronomical sample The new PDFS here presented can be tested on an astronomical sample representative of the IMF. Currently not all the various catalogs available on CDS report the column of the mass. As an example the promising IMF of IC 348 , see Figure 11 in Alves de Oliveira et al. (2012) , is not available on CDS.

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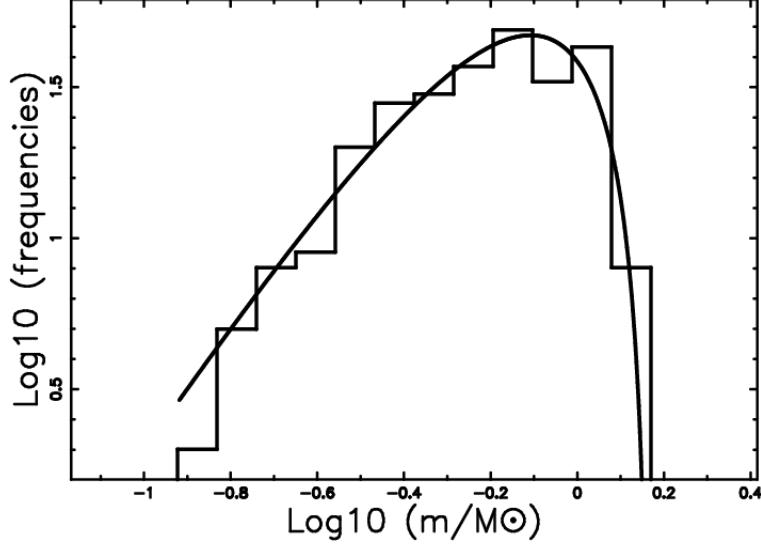


Fig. 1.— Logarithmic histogram of mass distribution as given by NGC 2362 cluster data (272 stars) with a superposition of the left truncated beta distribution when the number of bins, n , is 12, $a=0.12$, $b=1.47$, $\alpha=2.23$ and $\beta=3.09$. Vertical and horizontal axes have logarithmic scales.

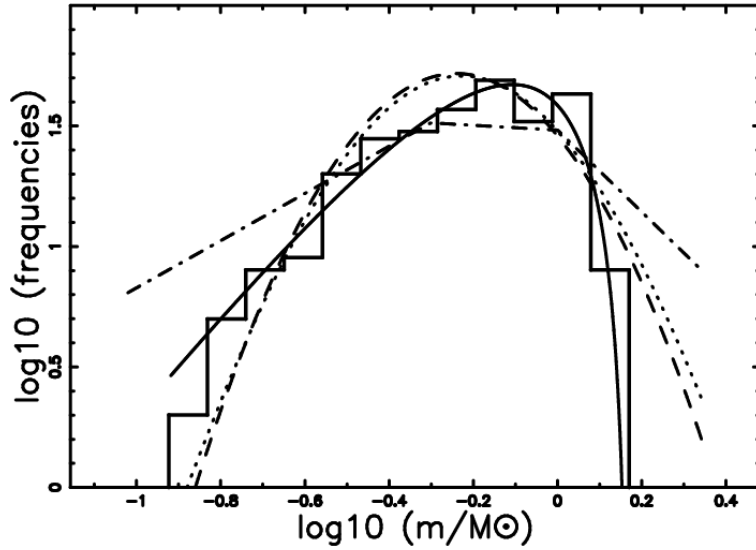


Fig. 2.— Histogram (step-diagram) of mass distribution as given by NGC 2362 cluster data (272 stars) with a superposition of the left truncated beta distribution (full line), the lognormal (dashed), the double Pareto lognormal (dotted) and the four power laws (dot-dash-dot-dash) . Vertical and horizontal axes have logarithmic scales.

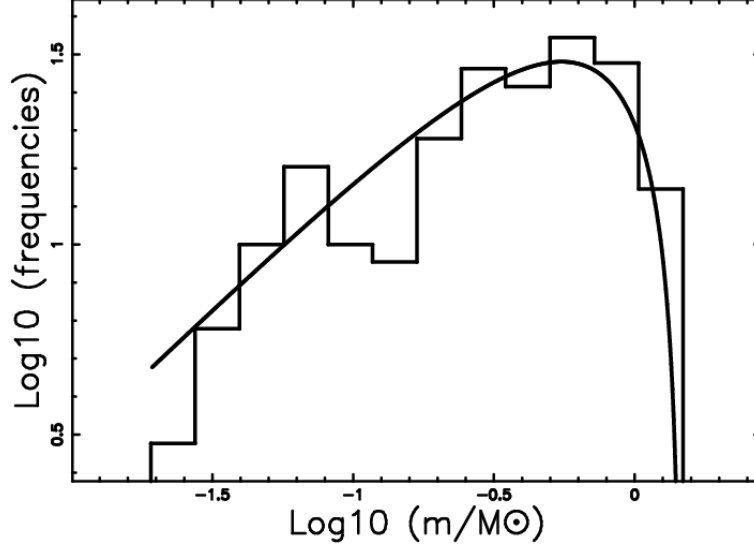


Fig. 3.— Logarithmic histogram of mass distribution as given by NGC 6611 cluster data (207 stars + BDs) with a superposition of the left truncated beta distribution when the number of bins, n , is 12, $a = 0.019$, $b = 1.46$, $\alpha = 0.55$ and $\beta = 1.6$. Vertical and horizontal axes have logarithmic scales.

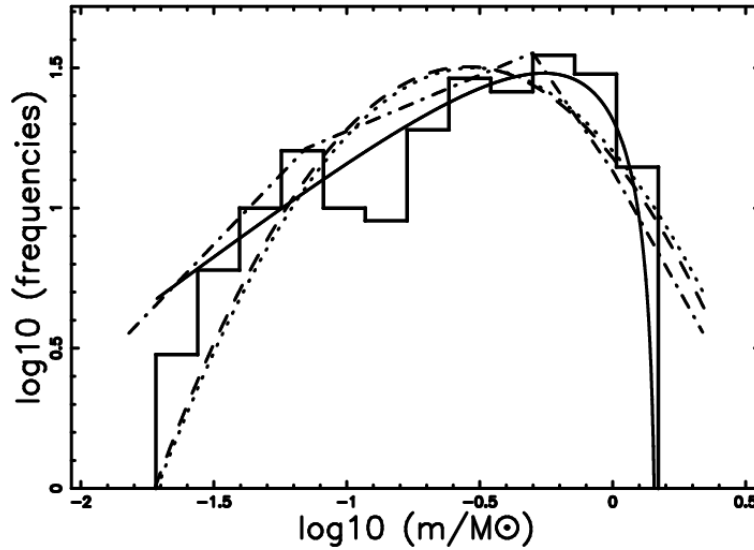


Fig. 4.— Histogram (step-diagram) of mass distribution as given by NGC 6611 cluster data (207 stars + BDs) with a superposition of the left truncated beta distribution (full line), the lognormal (dashed), the double Pareto lognormal (dotted) and the four power laws (dot-dash-dot-dash). Vertical and horizontal axes have logarithmic scales.

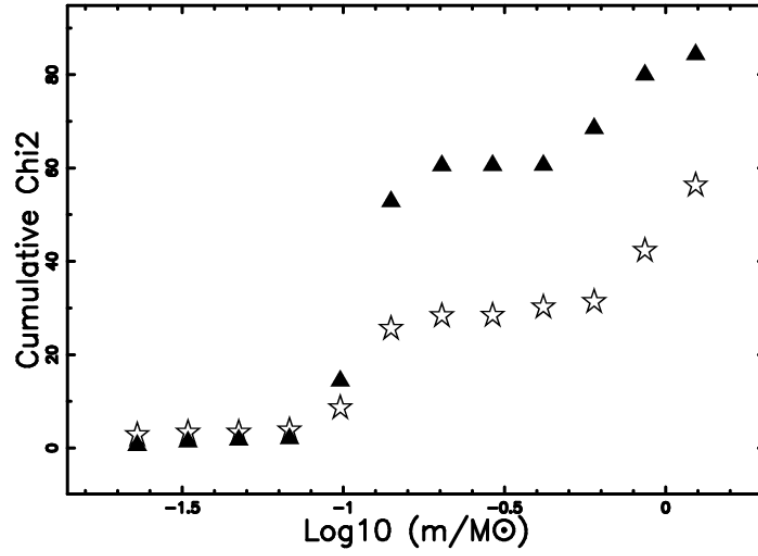


Fig. 5.— The residuals of the fits to NGC 6611 cluster data when 12 logarithmic bins are considered. The empty stars represent the left truncated beta PDF and the filled triangles the lognormal PDF.

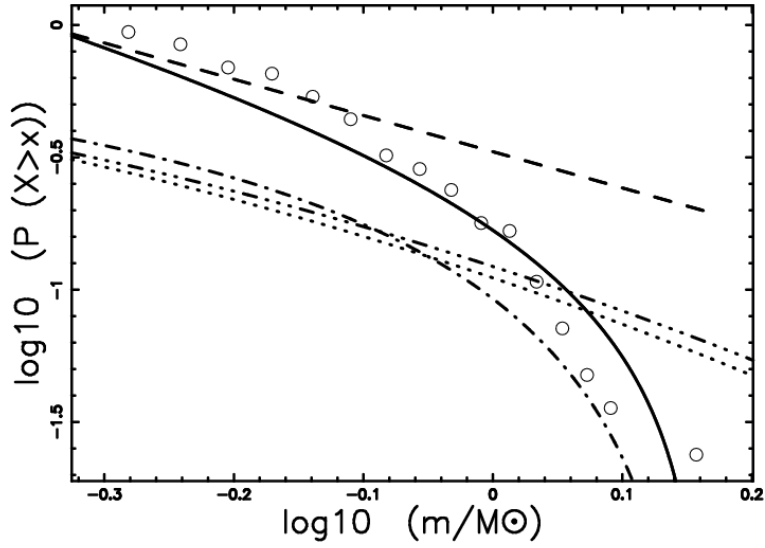


Fig. 6.— Survival function of NGC 6611 cluster data as $\log_{10} - \log_{10}$ plot when $M \geq 0.43M_{\odot}$: data (empty circles), survival function of the truncated Pareto pdf (full line) ($a=0.43, b=1.46, c=1.3$) and survival function of the Pareto pdf (dashed line) ($c=1.3$, Salpeter slope -2.3). The left truncated beta distribution (dot-dash-dot-dash), the lognormal (dotted) and the Double Pareto-lognormal (dash-dot-dot-dot) cover all the range in mass with parameters as in Table 2.